

Nijenhuis structures on Courant algebroids

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Abstract

We study Nijenhuis structures on Courant algebroids in terms of the canonical Poisson bracket on their symplectic realizations. We prove that the Nijenhuis torsion of a skew-symmetric endomorphism \mathcal{N} of a Courant algebroid is skew-symmetric if \mathcal{N}^2 is proportional to the identity, and only in this case when the Courant algebroid is irreducible. We derive a necessary and sufficient condition for a skew-symmetric endomorphism to give rise to a deformed Courant structure. In the case of the double of a Lie bialgebroid (A, A^*) , given an endomorphism N of A that defines a skew-symmetric endomorphism \mathcal{N} of the double of A , we prove that the torsion of \mathcal{N} is the sum of the torsion of N and that of the transpose of N .

Introduction

The aim of this paper is to study the infinitesimal deformations of Courant algebroids. We shall consider in detail the case of the double of a Lie bialgebroid, in particular the case of the double of a trivial Lie bialgebroid, such as the generalized tangent bundle of a manifold.

Nijenhuis operators for algebras other than Lie algebras were first considered by Fuchssteiner [11], while their role in the study of contractions of Lie algebras was first discussed by Bedjaoui-Tebbal [4]. The theory of Nijenhuis operators in the case of general algebraic structures was developed in [6] by Cariñena, Grabowski and Marmo, who identified the role they play in the theory of contractions and deformations of both Lie algebras and Leibniz (Loday) algebras. The case of Courant algebroids [24], which are important examples of Leibniz algebroids [14] and whose spaces of sections are therefore Leibniz algebras [23], was considered by them in [7], and then by Clemente-Gallardo and Nunes da Costa in [8], in both papers with applications to the deformation of Dirac structures. More recently, in [13],

Grabowski related the results obtained in [7] to Roytenberg's graded supermanifold approach to Courant algebroids [26], while Keller and Waldmann in [17] proceed by an alternative approach, using the Rothstein algebra in their study of the deformations of Courant algebroids. Nijenhuis tensors on the double of a Lie bialgebroid were also considered by Vaisman, in the case of the generalized tangent bundle $TM \oplus T^*M$ of a manifold, in his study of the reduction of generalized complex manifolds [29], by Stiénon and Xu in [28] and by Antunes in [1]. The relations of Nijenhuis structures on the double with Poisson-Nijenhuis structures have been discussed in these articles as well as in [7]. The deformation of generalized complex structures on manifolds was studied by Gualtieri in [15].

Our description of Nijenhuis structures and related concepts relies on the use of Roytenberg's graded Poisson bracket on the minimal symplectic realization of a Courant algebroid [26], and on its interplay with the big bracket [25] [18] [19] [20] when the Courant algebroid is the double of a Lie bialgebroid, and, in particular, in the case of the generalized tangent bundle of a manifold. We consider vector bundle endomorphisms that are skew-symmetric with respect to the fiberwise symmetric bilinear form of the Courant algebroid, a natural assumption that expresses the infinitesimal invariance of the symmetric bilinear form, and permits their inclusion in computations in the Poisson algebra of functions on the minimal symplectic realization. We argue that, in the deformation theory of a Courant structure, Θ , by a skew-symmetric tensor, \mathcal{N} , the decisive property is not the vanishing of the Nijenhuis torsion of \mathcal{N} but the property which we call 'weak deforming' (Definition 3.8). When \mathcal{N}^2 is a scalar multiple of the identity, a condition that appears repeatedly in [7] and in [8], this condition is equivalent to the 'weak Nijenhuis' condition introduced in [7], \mathcal{N} is a weak Nijenhuis tensor if the Nijenhuis torsion of \mathcal{N} is a cocycle for the differential $d_\Theta = \{\Theta, \cdot\}$. Our approach yields both new proofs of known results of [7] and [8], which we obtain with few or no computations, and several results which we believe have not appeared elsewhere, especially Theorems 3.9, 4.5 and 4.14.

In Section 1 we recall results of [6] and [7] on Nijenhuis structures on Leibniz algebras and Leibniz algebroids. In Section 2 we sketch the derived bracket approach to Courant algebroids [26] [17], and we give a definition of irreducibility, adapted from [13]. In Section 3, we study the properties of the Nijenhuis torsion of a skew-symmetric endomorphism, \mathcal{N} , of a Courant algebroid, and we show that the torsion has the usual properties of tensoriality and skew-symmetry in the special case where \mathcal{N}^2 is a scalar multiple of the identity (Theorem 3.6). In fact, on an irreducible Courant algebroid, any (skew-symmetric) Nijenhuis tensor is proportional to a complex,

paracomplex or subtangent structure (Theorem 3.7). We are thus led to a definition of ‘weak deforming tensors’ which are those tensors that generate infinitesimal deformations of Courant structures (Theorem 3.9).

Section 4 deals with those Courant algebroids that are the double of a Lie bialgebroid. In Theorem 4.5, we prove that, in the special case of an endomorphism, N , of a Lie algebroid, A , whose square is a scalar multiple of the identity, the torsion of the corresponding skew-symmetric endomorphism of its double $A \oplus A^*$ is, in a suitable sense, the sum of the torsion of N and the torsion of its transpose. Theorems 4.8 and 4.9 are reformulations or generalizations of results of [7] and [8]. Theorem 4.11 deals with the deformation of Lie bialgebroids. In particular, in the case of a trivial Lie bialgebroid, a Nijenhuis tensor on A defines a weak deforming tensor for $A \oplus A^*$ (Theorem 4.14). Finally, in Section 4.8, we outline the role of Poisson-Nijenhuis (or PN-) structures and of presymplectic-Nijenhuis (or Ω N-) structures on a Lie algebroid – for which see, e.g., [22] and references cited there – in the deformation theory of the double of the Lie algebroid. In Propositions 4.15 and 4.17, we prove that both PN-structures and Ω N-structures on a Lie algebroid, A , define infinitesimal deformations of the double $A \oplus A^*$ of A .

The role of the Nijenhuis tensors¹ in the theory of Dirac pairs [12] [9] [10] that generalize the bihamiltonian structures and have applications to

¹The early history of what is now called the Nijenhuis torsion of an endomorphism of the tangent bundle of a manifold is interesting and involved, and can be traced through Paulette Libermann’s article of 1955 in the *Bull. Soc. Math. France*, “Sur les structures presque complexes et autres structures infinitésimales régulières”. Charles Ehresmann defined the almost complex structures in 1947 and developed their study in his address to the ICM at Harvard, “Sur les variétés presque complexes” (1950). He defined the “torsion” so that its vanishing was a necessary condition for these structures to be integrable, and he proved its vanishing to be also sufficient in the real analytic case. (In the smooth case, the sufficiency was proved much later, in 1957, by Louis Nirenberg and A. Newlander.) This “torsion” was first defined as the torsion of a non canonically defined affine connexion associated with the almost complex structure. It follows from the results of Libermann’s doctoral thesis (1953) that this torsion is independent of the choice of the connexion and that its expression in local coordinates could be given in terms of the components of the tensorial field itself. There was related work, some published and some unpublished, by Beno Eckmann, Alfred Frölicher, Kentaro Nomizu, as well as by Eugenio Calabi, Georges de Rham, André Lichnerowicz and Jan A. Schouten. In her article, Libermann attributed to Eckmann the proof that the vanishing of this “torsion” is equivalent to the condition $[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$ for all tangent vector fields X and Y . The more general expression for the torsion of an endomorphism had been introduced by Albert Nijenhuis in 1951 in his study of the integrability of the distribution spanned by the eigenvectors of an endomorphism. This torsion was later recognized as a special case of the bracket introduced by Frölicher and Nijenhuis in their articles of 1956 and 1958.

integrable systems [3], and the theory of Dirac-Nijenhuis structures [7] [8] [16] will be the subject of further research.

1 Nijenhuis structures on Leibniz algebras

1.1 Leibniz algebras

A *Leibniz algebra* (or *Loday algebra*) is a vector space L over a field k of characteristic 0, equipped with a k -bilinear composition law, called the *Leibniz bracket*, satisfying the Jacobi identity,

$$u \circ (v \circ w) = (u \circ v) \circ w + v \circ (u \circ w), \quad (1)$$

for all u, v, w in L . A Leibniz algebra with a skew-symmetric composition law is a Lie algebra. The *Leibniz cohomology*, which was defined by Loday [23], is a generalization of the Chevalley-Eilenberg cohomology of Lie algebras.

1.2 Nijenhuis torsion

Let (L, \circ) be a Leibniz algebra. For an endomorphism N of L , define

$$u \circ_N v = Nu \circ v + u \circ Nv - N(u \circ v), \quad (2)$$

and set

$$(T_\circ N)(u, v) = Nu \circ Nv - N(u \circ_N v). \quad (3)$$

Then $T_\circ N : L \times L \rightarrow L$ is called the *Nijenhuis torsion* or simply the *torsion* of N , and N is said to be a *Nijenhuis tensor* or a *Nijenhuis structure* on (L, \circ) if $T_\circ N = 0$.

1.3 Deformations of Leibniz brackets

A necessary and sufficient condition for \circ_N to be a Leibniz bracket is that $T_\circ N$ be a Leibniz cocycle. Then \circ_N is a trivial infinitesimal deformation of \circ . In particular, Nijenhuis tensors define trivial infinitesimal deformations of Leibniz brackets. More precisely [7],

Proposition 1.1 *When N is a Nijenhuis tensor on (L, \circ) ,*

- (i) \circ_N is a Leibniz bracket,*
- (ii) N is a morphism of Leibniz algebras from (L, \circ_N) to (L, \circ) , and*
- (iii) \circ_N is compatible with \circ in the sense that their sum is a Leibniz bracket.*

1.4 Nijenhuis structures on Leibniz algebroids

Leibniz algebroids are generalizations of Lie algebroids in which the Lie bracket on the space of sections is only assumed to be a Leibniz bracket. They are defined in [14], where they are called *Loday algebroids*. The definitions of Nijenhuis torsion and Nijenhuis structures on Leibniz algebroids are similar to those in the case of Lie algebroids, since they involve only the bracket and do not make use of the anchors.

For a vector bundle endomorphism, $\mathcal{N} : E \rightarrow E$, of a Leibniz algebroid over a manifold M , we denote by the same letter, \mathcal{N} , the map it induces on the sections of E . Then we define the bracket $\circ_{\mathcal{N}}$ and the torsion of \mathcal{N} by formulas (2) and (3). A vector bundle endomorphism is called a *Nijenhuis tensor* or a *Nijenhuis structure* if its torsion vanishes.

2 Courant algebroids

2.1 The anchor and bracket as derived brackets

We follow the approach of Roytenberg [26]. Let $(E, \langle \cdot, \cdot \rangle)$ be a vector bundle equipped with a fiberwise symmetric bilinear form. Here we shall assume that $\langle \cdot, \cdot \rangle$ is non-degenerate. (Non-degeneracy is not assumed in the definition of Courant algebroids in [5] nor in that of Courant-Dorfman algebras in [27].) The minimal symplectic realization of $(E, \langle \cdot, \cdot \rangle)$ is the bundle $\tilde{E} = j^!(T^*[2]E[1])$, where $j : E[1] \rightarrow E[1] \oplus E^*[1]$ is defined by $u \mapsto (u, \frac{1}{2}\langle u, \cdot \rangle)$, and $j^!$ denotes the pull-back by j . The injective map, j , is such that $\langle ju, jv \rangle = \langle u, v \rangle$, for all $u, v \in E$, where $\langle \cdot, \cdot \rangle$ is the canonical fiberwise symmetric bilinear form on $E \oplus E^*$.

Let \mathcal{A} be the graded algebra of functions on the minimal symplectic realization \tilde{E} of E , equipped with its canonical Poisson bracket of degree -2 , which we denote by $\{ \cdot, \cdot \}$. Then, $\mathcal{A}^0 = C^\infty(M)$, $\mathcal{A}^1 = \Gamma E$, and for all sections u, v of E ,

$$\{u, v\} = \langle u, v \rangle. \quad (4)$$

A *Courant algebroid* structure on a vector bundle, E , over a manifold M , equipped with a fiberwise non-degenerate symmetric bilinear form, $\langle \cdot, \cdot \rangle$, is defined by an element $\Theta \in \mathcal{A}^3$ such that

$$\{\Theta, \Theta\} = 0. \quad (5)$$

The anchor, ρ , and the bracket, $[\cdot, \cdot]$, are defined by

$$\rho(u)f = \{ \{u, \Theta\}, f \}, \quad (6)$$

and

$$[u, v] = \{\{u, \Theta\}, v\}, \quad (7)$$

for all sections u, v of E , and $f \in C^\infty(M)$. Thus, the anchor and bracket are viewed as the derived brackets by Θ of the canonical Poisson bracket of \mathcal{A} restricted to $\mathcal{A}^1 \times \mathcal{A}^0 = \Gamma E \times C^\infty(M)$ and to $\mathcal{A}^1 \times \mathcal{A}^1 = \Gamma E \times \Gamma E$, respectively. Bracket $[\cdot, \cdot]$ is a Leibniz bracket on ΓE , called the *Dorfman bracket*. Courant algebroids are examples of Leibniz algebroids. The *Courant bracket* is the skew-symmetrization of the Dorfman bracket.

The operator $d_\Theta = \{\Theta, \cdot\}$ is a cohomology operator on \mathcal{A} .

We shall make use of the relations [26],

$$[u, v] + [v, u] = \partial \langle u, v \rangle, \quad (8)$$

and

$$\langle [u, v], w \rangle + \langle v, [u, w] \rangle = \langle u, \partial \langle v, w \rangle \rangle, \quad (9)$$

where $\partial : C^\infty(M) \rightarrow \Gamma E$ is defined by

$$\langle u, \partial f \rangle = \rho(u) \cdot f. \quad (10)$$

A vector bundle endomorphism ϕ of E is called *symmetric* if

$$\langle \phi u, v \rangle = \langle u, \phi v \rangle, \quad (11)$$

for all $u, v \in E$. This condition is written

$$\phi = {}^t\phi, \quad (12)$$

where ${}^t\phi$ is defined by $\langle \phi u, v \rangle = \langle u, {}^t\phi v \rangle$, for all sections u, v of E .

Before defining irreducible Courant algebroids, we consider the following properties, where ϕ is a vector bundle endomorphism of E ,
 (P_1) for all sections u and v of E ,

$$[u, \phi v] = \phi[u, v] \quad \text{and} \quad [\phi u, v] = \phi[u, v],$$

(P_2) for all sections u and v of E ,

$$[u, \phi v] = \phi[u, v] \quad \text{and} \quad \phi \partial \langle u, v \rangle = \partial \langle \phi u, v \rangle,$$

(P'_1) for all sections u and v of E ,

$$[u, \phi v] = \phi[u, v] \quad \text{and} \quad [\phi u, u] = \phi[u, u].$$

From relation (8) and the fact that the base field is not of characteristic 2, it is easy to prove the following lemma.

Lemma 2.1 *Let ϕ be a vector bundle endomorphism of $(E, \langle \cdot, \cdot \rangle)$. Properties (P_1) and (P_2) are equivalent. If ϕ is symmetric, properties (P_1) and (P'_1) are equivalent.*

We adopt the following definition:

Definition 2.2 *A Courant algebroid $(E, \langle \cdot, \cdot \rangle)$ is irreducible if any symmetric vector bundle endomorphism ϕ of E satisfying property (P_1) above is proportional to the identity endomorphism, Id_E , of E .*

Our definition is inspired by, but different from Grabowski's definition in [13] in which the endomorphisms are not required to be symmetric and irreducibility is defined by means of property (P'_1) . However, it follows from the lemma that any irreducible Courant algebroid in the sense of [13] is irreducible in the sense of Definition 2.2.

Examples of irreducible Courant algebroids will be given in Section 4.1.

2.2 Tensors on E and functions on \tilde{E}

Let $(E, \langle \cdot, \cdot \rangle)$ be a vector bundle equipped with a fiberwise non-degenerate symmetric bilinear form. Any tensor on E can be identified with a contravariant or a covariant tensor using the symmetric bilinear form, and any skew-symmetric contravariant or covariant tensor, t , can be identified with a function \tilde{t} on \tilde{E} . A skew-symmetric contravariant k -tensor t on E can be identified with a function on $E^*[1]$, and therefore also with a function \hat{t} on $E[1]$ by means of the symmetric bilinear form. The function \tilde{t} is the pull-back of \hat{t} under the projection $\tilde{E} \rightarrow E[1]$. We shall now describe these identifications by means of local coordinates.

Let (e_a) be a local basis of sections of E such that $\langle e_a, e_b \rangle$ is constant. Set $g_{ab} = \langle e_a, e_b \rangle$. If $(q^i, \tau^a, p_i, \theta_a)$ are local canonical coordinates on $T^*[2]E[1]$, then (q^i, τ^a, p_i) are local coordinates on \tilde{E} . Since, under the map j , $\theta_a = \frac{1}{2}g_{ab}\tau^b$, the non-vanishing Poisson brackets of these coordinates are

$$\{q^i, p_j\} = \delta_j^i \quad \text{and} \quad \{\tau^a, \tau^b\} = g^{ab}. \quad (13)$$

To a skew-symmetric contravariant k -tensor, $t = t^{a_1 a_2 \dots a_k} e_{a_1} e_{a_2} \dots e_{a_k}$, there corresponds

$$\tilde{t} = \frac{1}{k!} t^{a_1 a_2 \dots a_k} g_{a_1 b_1} g_{a_2 b_2} \dots g_{a_k b_k} \tau^{b_1} \tau^{b_2} \dots \tau^{b_k} \in \mathcal{A}^k. \quad (14)$$

When no confusion can arise, we shall sometimes write t for \tilde{t} . For instance, a section $u = u^a e_a$ of E is identified with the function $\tilde{u} = g_{ab} u^a \tau^b \in \mathcal{A}^1$.

Let $\mathcal{N} : E \rightarrow E$ be an endomorphism of the vector bundle E which preserves \langle , \rangle infinitesimally, i.e., such that

$$\langle \mathcal{N}u, v \rangle + \langle u, \mathcal{N}v \rangle = 0, \quad (15)$$

for all $u, v \in E$. This condition is written

$$\mathcal{N} + {}^t\mathcal{N} = 0. \quad (16)$$

Such a map is called *infinitesimally orthogonal* or *skew-symmetric*.

Remark 2.3 Other authors [7] [13] [1] call these endomorphisms *orthogonal*. In fact, when $\mathcal{N}^2 = \lambda \text{Id}_E$, condition $\mathcal{N} {}^t\mathcal{N} = \text{Id}_E$ is equivalent to $\mathcal{N} - \lambda {}^t\mathcal{N} = 0$. In particular, for a generalized almost complex structure, $\mathcal{N}^2 = -\text{Id}_E$, and the conditions $\mathcal{N} {}^t\mathcal{N} = \text{Id}_E$ and $\mathcal{N} + {}^t\mathcal{N} = 0$ are equivalent.

In local coordinates, if $\mathcal{N}(e_a) = \mathcal{N}_a^b e_b$, the condition $\mathcal{N} + {}^t\mathcal{N} = 0$ is $\mathcal{N}_a^b g_{bc} + \mathcal{N}_c^b g_{ba} = 0$. When it is satisfied, the associated contravariant tensor, with components $\mathcal{N}_c^a g^{cb}$, is skew-symmetric, and $\tilde{\mathcal{N}} = \frac{1}{2} \mathcal{N}_a^b g_{bc} \tau^a \tau^c \in \mathcal{A}^2$. A short computation shows that

$$\mathcal{N}(u) = \{u, \tilde{\mathcal{N}}\}, \quad (17)$$

for all sections u of E . In fact, when \mathcal{N} is a skew-symmetric endomorphism, $\tilde{\mathcal{N}}$ is the unique element in \mathcal{A}^2 satisfying (17). See [2] for more general results on the correspondence between tensors on E and functions on \tilde{E} .

3 Nijenhuis and deforming tensors on Courant algebroids

Let $(E, \langle , \rangle, \Theta)$ be a Courant algebroid over a manifold M , where \langle , \rangle is the fiberwise non-degenerate symmetric bilinear form, and $\Theta \in \mathcal{A}^3$ determines the anchor, ρ , and the Leibniz bracket on sections, $[,]$.

3.1 Nijenhuis torsion

In what follows, we shall assume that $\mathcal{N} : E \rightarrow E$ is a skew-symmetric vector bundle endomorphism. This is a natural assumption since skew-symmetry means that \mathcal{N} leaves \langle , \rangle infinitesimally invariant. As above, we define

$$[u, v]_{\mathcal{N}} = [\mathcal{N}u, v] + [u, \mathcal{N}v] - \mathcal{N}[u, v]. \quad (18)$$

Lemma 3.1 *In terms of the Poisson bracket of \mathcal{A} ,*

$$[u, v]_{\mathcal{N}} = \{\{u, \{\tilde{\mathcal{N}}, \Theta\}\}, v\}, \quad (19)$$

for all $u, v \in \Gamma E \simeq \mathcal{A}^1$.

Proof The proof is an application of the Jacobi identity for the Poisson bracket, formally identical to the proof of the analogous formula for Lie algebroids. See, e.g., lemma 2 of [22]. \square

We now define the *Nijenhuis torsion*, or simply the *torsion*, $T_{\Theta}\mathcal{N}$, of \mathcal{N} , as in (3), by

$$(T_{\Theta}\mathcal{N})(u, v) = [\mathcal{N}u, \mathcal{N}v] - \mathcal{N}[u, v]_{\mathcal{N}}, \quad (20)$$

for all sections u, v of E . A skew-symmetric endomorphism whose torsion vanishes is called a *Nijenhuis tensor*.

Remark 3.2 For an endomorphism that is not skew-symmetric, the torsion can still be defined by (18) and (20), and we observe that, if $\mathcal{N}' = \mathcal{N} + \kappa \text{Id}_E$, where κ is a scalar, then $T_{\Theta}(\mathcal{N}') = T_{\Theta}\mathcal{N}$. Thus, those properties of the torsion that are proved under the assumption that \mathcal{N} is skew-symmetric but whose proof does not utilize the Poisson bracket are also valid for endomorphisms $\mathcal{N}' = \mathcal{N} + \kappa \text{Id}_E$, which are characterized by the condition $\mathcal{N}' + {}^t\mathcal{N}' = 2\kappa \text{Id}_E$. Such endomorphisms are called *paired* in [7].

Remark 3.3 One can also define the torsion $T_C\mathcal{N}$ of an endomorphism \mathcal{N} with respect to the Courant bracket, $[\cdot, \cdot]_C$, replacing the Dorfman bracket by its skew-symmetrization in the preceding formulas. The relation between the two torsions is

$$(T_C\mathcal{N})(u, v) = \frac{1}{2}((T_{\Theta}\mathcal{N})(u, v) - (T_{\Theta}\mathcal{N})(v, u)), \quad (21)$$

while, for a skew-symmetric tensor \mathcal{N} ,

$$\begin{aligned} & (T_C\mathcal{N})(u, v) - (T_{\Theta}\mathcal{N})(u, v) \\ &= \frac{1}{2}(-\partial\langle \mathcal{N}u, \mathcal{N}v \rangle + \mathcal{N}\partial\langle \mathcal{N}u, v \rangle + \mathcal{N}\partial\langle u, \mathcal{N}v \rangle - \mathcal{N}^2\partial\langle u, v \rangle) \\ &= \frac{1}{2}(\partial\langle u, \mathcal{N}^2v \rangle - \mathcal{N}^2\partial\langle u, v \rangle). \end{aligned} \quad (22)$$

If \mathcal{N}^2 is a scalar multiple of the identity of E , both torsions, $T_{\Theta}\mathcal{N}$ and $T_C\mathcal{N}$, coincide.

3.2 Properties of the torsion

For ease of exposition, we introduce the following definition from [29] (see also [1]), where ‘cps’ stands for ‘complex, paracomplex or subtangential’.

Definition 3.4 *A skew-symmetric endomorphism \mathcal{N} of a Courant algebroid E such that $\mathcal{N}^2 = \lambda \text{Id}_E$, where $\lambda = -1, 1$ or 0 , is called a generalized almost cps structure on E . A generalized almost cps structure is called a generalized cps structure if its torsion vanishes.*

When $(E, \langle \cdot, \cdot \rangle, \Theta)$ is a Courant algebroid, the torsion $T_\Theta \mathcal{N}$ of a skew-symmetric endomorphism \mathcal{N} of E is a map from $\Gamma E \times \Gamma E$ to ΓE . Unlike the case of Lie algebroids, $T_\Theta \mathcal{N}$ is not in general $C^\infty(M)$ -linear in both arguments, and in general not skew-symmetric.

Linearity. It is clear that

$$[u, fv] = f[u, v] + (\rho(u) \cdot f)v \quad (23)$$

and

$$[fu, v] = -[v, fu] + \partial \langle fu, v \rangle = f[u, v] - (\rho(v) \cdot f)u + \langle u, v \rangle \partial f. \quad (24)$$

Whence,

$$(T_\Theta \mathcal{N})(u, fv) = f(T_\Theta \mathcal{N})(u, v), \quad (25)$$

and

$$(T_\Theta \mathcal{N})(fu, v) = f(T_\Theta \mathcal{N})(u, v) + \langle u, v \rangle \mathcal{N}^2(\partial f) - \langle u, \mathcal{N}^2 v \rangle \partial f. \quad (26)$$

In fact, since \mathcal{N} is skew-symmetric,

$$\begin{aligned} & (T_\Theta \mathcal{N})(fu, v) - f(T_\Theta \mathcal{N})(u, v) \\ &= (\langle \mathcal{N}u, v \rangle + \langle u, \mathcal{N}v \rangle) \partial f + \langle \mathcal{N}u, \mathcal{N}v \rangle \partial f + \langle u, v \rangle \mathcal{N}^2(\partial f) \\ &= \langle u, v \rangle \mathcal{N}^2(\partial f) - \langle u, \mathcal{N}^2 v \rangle \partial f. \end{aligned}$$

Skew-symmetry. Again using the fact that \mathcal{N} is skew-symmetric, we obtain

$$(T_\Theta \mathcal{N})(u, v) + (T_\Theta \mathcal{N})(v, u) = \mathcal{N}^2 \partial \langle u, v \rangle - \partial \langle u, \mathcal{N}^2 v \rangle \quad (27)$$

since

$$(T_\Theta \mathcal{N})(u, v) + (T_\Theta \mathcal{N})(v, u) = \partial \langle \mathcal{N}u, \mathcal{N}v \rangle - \mathcal{N}(\partial \langle \mathcal{N}u, v \rangle + \partial \langle u, \mathcal{N}v \rangle) + \mathcal{N}^2 \partial \langle u, v \rangle.$$

Remark 3.5 Equation (27) can alternatively be derived from (21) and (22).

Associated 3-tensor. In order to determine whether $T_\Theta \mathcal{N}$ determines a skew-symmetric covariant 3-tensor, we use the skew-symmetry of \mathcal{N} and relation (9) to obtain

$$\langle (T_\Theta \mathcal{N})(u, v), w \rangle + \langle (T_\Theta \mathcal{N})(u, w), v \rangle = \langle \mathcal{N}^2[u, w] - [u, \mathcal{N}^2 w], v \rangle. \quad (28)$$

Equations (26), (27) and (28) show that, when $\mathcal{N}^2 = \lambda \text{Id}_E$, the torsion $T_\Theta \mathcal{N}$ of \mathcal{N} is $C^\infty(M)$ -linear in both arguments and skew-symmetric, and defines a skew-symmetric covariant 3-tensor, $\widetilde{T_\Theta \mathcal{N}}$, on E by

$$\widetilde{T_\Theta \mathcal{N}}(u, v, w) = \langle (T_\Theta \mathcal{N})(u, v), w \rangle. \quad (29)$$

More precisely,

Theorem 3.6 *Assume that \mathcal{N} is proportional to a generalized almost cps structure on a Courant algebroid, $(E, \langle \cdot, \cdot \rangle, \Theta)$.*

(i) The torsion, $T_\Theta \mathcal{N}$, of \mathcal{N} is $C^\infty(M)$ -linear in both arguments and skew-symmetric, and it defines an element $\widetilde{T_\Theta \mathcal{N}} \in \mathcal{A}^3$.

(ii) For all sections u, v of E ,

$$(T_\Theta \mathcal{N})(u, v) = \{ \{ u, \widetilde{T_\Theta \mathcal{N}} \}, v \}. \quad (30)$$

(iii) Set $\mathcal{N}^2 = \lambda \text{Id}_E$, for a real number λ . Then

$$\widetilde{T_\Theta \mathcal{N}} = -\frac{1}{2}(\{ \{ \tilde{\mathcal{N}}, \Theta \}, \tilde{\mathcal{N}} \} + \lambda \Theta). \quad (31)$$

Proof Formulas (30) and (31) follow from (17) and (19), and the use of the Jacobi identity for $\{ \cdot, \cdot \}$. \square

In addition, in view of Definition 2.2 and Lemma 2.1, from relations (28) and (27), we obtain immediately,

Theorem 3.7 *If \mathcal{N} is a Nijenhuis tensor on E , then $\mathcal{N}^2[u, v] = [u, \mathcal{N}^2 v]$ and $\mathcal{N}^2 \partial \langle u, v \rangle = \partial \langle u, \mathcal{N}^2 v \rangle$, for all sections u, v of E . If E is irreducible, any Nijenhuis tensor on E is proportional to a generalized cps structure.*

Formula (31) was first stated in corollary 3 of [13]. A result equivalent to Theorem 3.7 was proved in [7] (theorem 5).

3.3 Deformations of Courant algebroids

As above, we shall consider skew-symmetric endomorphisms \mathcal{N} of $(E, \langle \cdot, \cdot \rangle)$ exclusively. In fact, tensors with vanishing Nijenhuis torsion do not in general define trivial infinitesimal deformations of the Dorfman bracket of a Courant algebroid, unless they have additional properties such as being proportional to a generalized almost cps structure. We are thus led to introduce the following definitions.

Definition 3.8 *A skew-symmetric endomorphism \mathcal{N} of a Courant algebroid $(E, \langle \cdot, \cdot \rangle, \Theta)$ is called a*

- (i) *weak deforming tensor for Θ if $\{\{\tilde{\mathcal{N}}, \Theta\}, \tilde{\mathcal{N}}\}$ is a d_Θ -cocycle,*
- (ii) *deforming tensor for Θ if $\{\{\tilde{\mathcal{N}}, \Theta\}, \tilde{\mathcal{N}}\}$ is a scalar multiple of Θ ,*

This terminology is justified by the fact that, because $d_\Theta \Theta = \{\Theta, \Theta\} = 0$, any deforming tensor is a weak deforming tensor. Theorem 3.9 below is further justification for the terms that we have introduced.

Theorem 3.9 *Let \mathcal{N} be a skew-symmetric endomorphism of a Courant algebroid $(E, \langle \cdot, \cdot \rangle, \Theta)$. Then $\{\tilde{\mathcal{N}}, \Theta\}$ is a Courant algebroid structure on $(E, \langle \cdot, \cdot \rangle)$ if and only if \mathcal{N} is a weak deforming tensor for Θ .*

Proof The theorem follows from the fact that, by the Jacobi identity,

$$\{\{\tilde{\mathcal{N}}, \Theta\}, \{\tilde{\mathcal{N}}, \Theta\}\} = \{\Theta, \{\{\tilde{\mathcal{N}}, \Theta\}, \tilde{\mathcal{N}}\}\},$$

so $\{\tilde{\mathcal{N}}, \Theta\}$ is a Courant algebroid structure on $(E, \langle \cdot, \cdot \rangle)$ if and only if $\{\{\tilde{\mathcal{N}}, \Theta\}, \tilde{\mathcal{N}}\}$ is a d_Θ -cocycle. \square

Remark 3.10 The condition $\{\{\tilde{\mathcal{N}}, \Theta\}, \tilde{\mathcal{N}}\} = 0$ is sufficient for $\{\tilde{\mathcal{N}}, \Theta\}$ to be a Courant algebroid structure. It expresses the vanishing of the Maurer-Cartan element $[\tilde{\mathcal{N}}, \tilde{\mathcal{N}}]^\Theta$ in the differential graded Leibniz-Poisson algebra $(\mathcal{A}, [\cdot, \cdot]^\Theta, 0)$, where $[\cdot, \cdot]^\Theta$ is the derived bracket of the Poisson bracket $\{\cdot, \cdot\}$ by the odd interior derivation of square 0, $\{\Theta, \cdot\}$.

When \mathcal{N} is a weak deforming tensor, the Courant algebroid structure $\{\tilde{\mathcal{N}}, \Theta\}$ on $(E, \langle \cdot, \cdot \rangle)$ is compatible with Θ . In fact $\{\Theta + \{\tilde{\mathcal{N}}, \Theta\}, \Theta + \{\tilde{\mathcal{N}}, \Theta\}\}$ vanishes.

Now, the condition $\{\Theta, T_\Theta \mathcal{N}\} = 0$ makes sense only if $T_\Theta \mathcal{N}$ is an element of \mathcal{A}^3 . If E is irreducible this is the case if and only if \mathcal{N} is proportional to a generalized almost cps structure, whence the following definition.

Definition 3.11 *If \mathcal{N} is proportional to a generalized almost cps structure and if $T_\Theta \mathcal{N}$ is a d_Θ -cocycle, \mathcal{N} is called a weak Nijenhuis tensor.*

In the case of tensors proportional to a generalized almost cps structure, we observe the following implications and equivalence:

- A Nijenhuis tensor is a weak Nijenhuis tensor.
- A Nijenhuis tensor is a deforming tensor, and therefore also a weak deforming tensor.
- A tensor is weak Nijenhuis if and only if it is weak deforming.

$$\begin{array}{ccc} \text{Nijenhuis} & \Rightarrow & \text{weak Nijenhuis} \\ \Downarrow & & \Downarrow \\ \text{deforming} & \Rightarrow & \text{weak deforming} \end{array}$$

We can now state a corollary of Theorem 3.9 concerning the special case of those endomorphisms whose square is a scalar multiple of the identity.

Corollary 3.12 *Let \mathcal{N} be a skew-symmetric endomorphism of a Courant algebroid $(E, \langle \cdot, \cdot \rangle, \Theta)$, proportional to a generalized almost cps structure. Then $\{\tilde{\mathcal{N}}, \Theta\}$ is a Courant algebroid structure on $(E, \langle \cdot, \cdot \rangle)$ if and only if \mathcal{N} is a weak Nijenhuis tensor.*

While the compatibility of Θ and $\{\tilde{\mathcal{N}}, \Theta\}$ is satisfied as soon as \mathcal{N} is weak deforming, it is the vanishing of the torsion which implies a morphism property of \mathcal{N} . If we recall that a generalized almost cps structure is a generalized cps structure if and only if its torsion vanishes, we can state,

Proposition 3.13 *Let \mathcal{N} be a skew-symmetric endomorphism of E proportional to a generalized almost cps structure. Then \mathcal{N} is a morphism of Courant algebroids from $(E, \langle \cdot, \cdot \rangle, \{\tilde{\mathcal{N}}, \Theta\})$ to $(E, \langle \cdot, \cdot \rangle, \Theta)$ if and only if \mathcal{N} is proportional to a generalized cps structure.*

Corollary 3.12 and Proposition 3.13 imply and are implied by results to be found in [7] and [13].

4 The case of the double of a Lie bialgebroid

4.1 The double of a Lie bialgebroid

Let A be a vector bundle and let $E = A \oplus A^*$ be equipped with the canonical symmetric bilinear form $\langle \cdot, \cdot \rangle$. The minimal symplectic realization of E is

$\tilde{E} = T^*[2]A[1]$, and the canonical Poisson bracket of \mathcal{A} coincides with the big bracket (for which see [25] [19] [20]), which we also denote by $\{ , \}$.

Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid over a manifold M . Then μ and γ are elements of \mathcal{A}^3 satisfying $\{\mu, \mu\} = \{\mu, \gamma\} = \{\gamma, \gamma\} = 0$. The canonical symmetric bilinear form and $\Theta = \mu + \gamma$ turn $E = A \oplus A^*$ into a Courant algebroid called the *double* of $((A, \mu), (A^*, \gamma))$. See [24]. In Section 4.7, we shall consider the case $\gamma = 0$, in which case the Lie bialgebroid is called *trivial*. In particular if $A = TM$ equipped with the identity endomorphism as anchor and the Lie bracket of vector fields and if $\gamma = 0$, then $TM \oplus T^*M$ is the *standard Courant algebroid* or *generalized tangent bundle* of M .

Definition 4.1 *A Lie algebroid A is called irreducible if any vector bundle endomorphism ψ of A satisfying $\psi[X, Y] = [X, \psi Y]$, for all sections X, Y of A , is proportional to the identity, Id_A , of A .*

It is proved in [7] that the tangent bundle of any connected manifold is an irreducible Lie algebroid. We can now give examples of irreducible Courant algebroids.

When (A, μ) is an irreducible Lie algebroid over a connected manifold, the double of the trivial Lie bialgebroid $((A, \mu), (A^*, 0))$ is an irreducible Courant algebroid in the sense of Definition 2.2. In particular, the generalized tangent bundle of a connected manifold is an irreducible Courant algebroid. To prove this statement, we write a symmetric endomorphism of $A \oplus A^*$ as $\phi = \begin{pmatrix} \psi & \alpha \\ \beta & {}^t\psi \end{pmatrix}$, where ψ is an endomorphism of A , and $\alpha : A^* \rightarrow A$ and $\beta : A \rightarrow A^*$ are symmetric. We then express the conditions $\phi[u, v] = [\phi u, v]$ and $\phi[u, v] = [u, \phi v]$ for, successively, $u = X$ and $v = Y$, then $u = X$ and $v = \eta$, then $u = \xi$ and $v = Y$, then $u = \xi$ and $v = \eta$, where $X, Y \in \Gamma A$ and $\xi, \eta \in \Gamma(A^*)$. By the irreducibility of (A, μ) , we find that ψ is a constant multiple of the identity of A , then that α and β must vanish.

4.2 Generalized almost cps structures on $A \oplus A^*$

Any vector bundle endomorphism of $E = A \oplus A^*$ is of the form $\mathcal{N} = \begin{pmatrix} N & \pi \\ \omega & N' \end{pmatrix}$, where $N : A \rightarrow A$, $N' : A^* \rightarrow A^*$, $\pi : A^* \rightarrow A$ and $\omega : A \rightarrow A^*$. The endomorphism \mathcal{N} is skew-symmetric if and only if $N' = -{}^tN$, π is a bivector on A , and ω is a 2-form on A .

The conditions for $\mathcal{N}^2 = \lambda \text{Id}_E$ are (i) $N\pi$ is a bivector, (ii) ωN is a 2-form and (iii) $N^2 + \pi\omega = \lambda \text{Id}_A$. A sufficient condition for (iii) is that

N^2 be a scalar multiple of the identity and that π and ω be inverses of one another, or that $\pi = 0$, or that $\omega = 0$. See [28] [1] for the general case and its interpretation in terms of Poisson quasi-Nijenhuis structures.

4.3 Tensors on A and functions on $T^*[2]A[1]$

Let A be a vector bundle. We show that skew-symmetric tensors on A can be identified with elements of \mathcal{A} , the graded algebra of functions on $T^*[2]A[1]$.

A tensor $t \in A^* \otimes A$ can be considered as an element in $(A \oplus A^*) \otimes (A \oplus A^*)$ by setting

$$t(X + \xi; Y + \eta) = \langle t(X), \eta \rangle, \quad (32)$$

and, because $A \oplus A^*$ is self-dual, t can be skew-symmetrized into the element \tilde{t} in $\wedge^2(A \oplus A^*)$ such that

$$\tilde{t}(X + \xi, Y + \eta) = \langle t(X), \eta \rangle - \langle t(Y), \xi \rangle, \quad (33)$$

for all $X, Y \in A$ and $\xi, \eta \in A^*$. The map induced on sections of $A \oplus A^*$ by \tilde{t} is the element in \mathcal{A} that corresponds to t .

In other words, if N is a vector bundle endomorphism of A , considered as an element in $A^* \otimes A$, then the skew-symmetric endomorphism \mathcal{N} of $A \oplus A^*$ defined by N is such that

$$\mathcal{N}(X + \xi) = NX - {}^tN\xi, \quad (34)$$

and as in (17),

$$\mathcal{N}(X + \xi) = \{X + \xi, \tilde{\mathcal{N}}\}. \quad (35)$$

We can also skew-symmetrize higher-order tensors. A tensor $t \in \wedge^2 A^* \otimes A$ can be considered as an element in $\wedge^2(A \oplus A^*) \otimes (A \oplus A^*)$ by setting

$$t(X + \xi, Y + \eta; Z + \zeta) = \langle t(X, Y), \zeta \rangle, \quad (36)$$

in which case \tilde{t} is the element in $\wedge^3(A \oplus A^*)$ such that

$$\tilde{t}(X + \xi, Y + \eta, Z + \zeta) = \langle t(X, Y), \zeta \rangle + \langle t(Y, Z), \xi \rangle + \langle t(Z, X), \eta \rangle, \quad (37)$$

for all $X, Y, Z \in A$ and $\xi, \eta, \zeta \in A^*$. The map induced on sections of $A \oplus A^*$ by \tilde{t} is the element in \mathcal{A} that corresponds to t . Then

$$\tilde{t}(X + \xi, Y + \eta, Z + \zeta) = \{\{\{X + \xi, \tilde{t}\}, Y + \eta\}, Z + \zeta\}. \quad (38)$$

Similarly, if $t \in \wedge^2 A \otimes A^*$, then $\tilde{t} \in \wedge^3(A \oplus A^*)$ is defined by

$$\tilde{t}(X + \xi, Y + \eta, Z + \zeta) = \langle t(\xi, \eta), Z \rangle + \langle t(\eta, \zeta), X \rangle + \langle t(\zeta, \xi), Y \rangle. \quad (39)$$

Let (e_α) be a local basis of sections of A and let (ϵ^α) be the dual basis. Let $(x^i, \tau^\alpha, p_i, \theta_\alpha)$ be local canonical coordinates on $T^*[2]A[1]$. If $N = N_\beta^\alpha \epsilon^\beta e_\alpha$, then $\tilde{N} = N_\beta^\alpha \tau^\beta \theta_\alpha$. If $t = \frac{1}{2} t_{\beta\gamma}^\alpha \epsilon^\beta \epsilon^\gamma e_\alpha$, then $\tilde{t} = \frac{1}{2} t_{\beta\gamma}^\alpha \tau^\beta \tau^\gamma \theta_\alpha$. As elements in \mathcal{A}^2 , N and \tilde{N} are identified, and as elements in \mathcal{A}^3 , t and \tilde{t} are identified.

4.4 The double of deformed brackets

Let (A, μ) and (A^*, γ) be Lie algebroids, with brackets denoted by $[\cdot, \cdot]^\mu$ and $[\cdot, \cdot]^\gamma$. Let $N : A \rightarrow A$ be a vector bundle endomorphism. Set $\{N, \mu\} = \mu_N$ and $\{N, \gamma\} = \gamma_{-t_N}$. The associated brackets are $[\cdot, \cdot]^{\mu_N} = [\cdot, \cdot]_N^\mu$ on A , and $[\cdot, \cdot]^{\gamma_{-t_N}} = [\cdot, \cdot]_{-t_N}^\gamma$ on A^* , which are not necessarily Lie brackets. We consider the bracket on $A \oplus A^*$ defined by $\mu + \gamma$ (respectively, $\mu_N + \gamma_{-t_N}$), which we call the *double* of brackets μ and γ (respectively, $[\cdot, \cdot]_N^\mu$ and $[\cdot, \cdot]_{-t_N}^\gamma$).

Proposition 4.2 *Let (A, μ) and (A^*, γ) be Lie algebroids and let $[\cdot, \cdot]^{\mu+\gamma}$ be the double of brackets $[\cdot, \cdot]^\mu$ and $[\cdot, \cdot]^\gamma$ in $A \oplus A^*$. Let N be a vector bundle endomorphism of A , and let \mathcal{N} be the associated skew-symmetric endomorphism of $A \oplus A^*$ defined by (34). Then bracket $[\cdot, \cdot]_{\mathcal{N}}^{\mu+\gamma}$ is the double of brackets $[\cdot, \cdot]_N^\mu$ and $[\cdot, \cdot]_{-t_N}^\gamma$.*

Proof It is clear that

$$\{\tilde{\mathcal{N}}, \mu + \gamma\} = \{\tilde{\mathcal{N}}, \mu\} + \{\tilde{\mathcal{N}}, \gamma\} = \{N, \mu\} + \{N, \gamma\}. \quad (40)$$

The result follows. \square

Remark 4.3 The result of the proposition is valid more generally, independently of the assumptions $\{\mu, \mu\} = 0$ and $\{\gamma, \gamma\} = 0$ which express the fact that (A, μ) and (A^*, γ) are Lie algebroids.

4.5 Torsion of \mathcal{N} in the case of the double of a Lie bialgebroid

When (A, μ) and (A^*, γ) are Lie algebroids, if the torsion $T_{\mu+\gamma}\mathcal{N}$ of \mathcal{N} defines an element in \mathcal{A}^3 , we can compare $\widetilde{T_{\mu+\gamma}\mathcal{N}}$ with the sum of the elements in $\Gamma(\wedge^3(A \oplus A^*)) \subset \mathcal{A}^3$ defined by the torsion $T_\mu N$ of N and the torsion $T_\gamma {}^tN$ of tN .

Definition 4.4 *A vector bundle endomorphism N of a vector bundle A is called an almost cps structure if $N^2 = \epsilon \text{Id}_A$, with $\epsilon = -1, 1$ or 0 . On a Lie algebroid (A, μ) , an almost cps structure is called a cps structure if, in addition, the Nijenhuis torsion $T_\mu N$ of N vanishes.*

Theorem 4.5 *Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid. Let $N : A \rightarrow A$ be a vector bundle endomorphism, and let \mathcal{N} be the skew-symmetric endomorphism of $A \oplus A^*$ with matrix $\begin{pmatrix} N & 0 \\ 0 & {}^tN \end{pmatrix}$.*

- (i) *The element $\{\{\tilde{\mathcal{N}}, \mu + \gamma\}, \tilde{\mathcal{N}}\}$ is in \mathcal{A}^3 and is equal to $\{\{N, \mu + \gamma\}, N\}$.*
(ii) *If N is proportional to an almost cps structure on A , then*

$$\widetilde{T_{\mu+\gamma}\mathcal{N}} = \widetilde{T_\mu N} + \widetilde{T_\gamma {}^tN}. \quad (41)$$

The explicit form of equation (41) is

$$\begin{aligned} & (T_{\mu+\gamma}\mathcal{N})(X + \xi, Y + \eta, Z + \zeta) \\ &= (T_\mu N)(X, Y, \zeta) + (T_\mu N)(Y, Z, \xi) + (T_\mu N)(Z, X, \eta) \\ &+ (T_\gamma {}^tN)(\xi, \eta, Z) + (T_\gamma {}^tN)(\eta, \zeta, X) + (T_\gamma {}^tN)(\zeta, \xi, Y), \end{aligned} \quad (42)$$

for all sections $X + \xi, Y + \eta, Z + \zeta$ of $A \oplus A^$.*

Proof The proof of (i) is based on the remarks that $\{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\}$ (respectively, $\{\{\tilde{\mathcal{N}}, \gamma\}, \tilde{\mathcal{N}}\}$) is equal to the element $\{\{N, \mu\}, N\}$ (respectively, $\{\{N, \gamma\}, N\}$) of \mathcal{A}^3 . Therefore, by Proposition 4.2, $\{\{\tilde{\mathcal{N}}, \mu + \gamma\}, \tilde{\mathcal{N}}\}$ is equal to $\{\{N, \mu + \gamma\}, N\}$.

Since, when $N^2 = \lambda \text{Id}_A$, by formula (31),

$$T_\mu N = -\frac{1}{2}(\{\{N, \mu\}, N\} + \lambda\mu), \quad T_\gamma {}^tN = -\frac{1}{2}(\{\{N, \gamma\}, N\} + \lambda\gamma),$$

and

$$\widetilde{T_{\mu+\gamma}\mathcal{N}} = -\frac{1}{2}(\{\{\tilde{\mathcal{N}}, \mu + \gamma\}, \tilde{\mathcal{N}}\} + \lambda(\mu + \gamma)),$$

the result of (ii) follows. \square

Remark 4.6 The result of the theorem is valid more generally, independently of the assumptions $\{\mu, \mu\} = \{\gamma, \gamma\} = \{\mu, \gamma\} = 0$ which express the fact that $((A, \mu), (A^*, \gamma))$ is a Lie bialgebroid.

Remark 4.7 Since the Dorfman bracket on $\Gamma(A \oplus A^*)$ reduces to $[\cdot, \cdot]^\mu$ on ΓA and to $[\cdot, \cdot]^\gamma$ on ΓA^* , it is clear that, for any endomorphism N of A ,

$$(T_{\mu+\gamma}\mathcal{N})|_A = T_\mu N \quad \text{and} \quad (T_{\mu+\gamma}\mathcal{N})|_{A^*} = T_\gamma {}^tN. \quad (43)$$

As a consequence of Theorem 4.5 (ii), we obtain,

Theorem 4.8 *Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid. Let N be proportional to an almost cps structure on A , and assume that N is a Nijenhuis tensor for (A, μ) and (A^*, γ) , i.e., $T_\mu N = 0$ and $T_\gamma {}^t N = 0$. Then N gives rise to a Nijenhuis tensor $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^t N \end{pmatrix}$ for the Courant algebroid $(A \oplus A^*, \mu + \gamma)$.*

For any scalar κ , $T_\mu(N + \kappa \text{Id}_A) = T_\mu N$, $T_\gamma({}^t N + \kappa \text{Id}_{A^*}) = T_\gamma({}^t N)$ and $T_{\mu+\gamma}(\mathcal{N} + \kappa \text{Id}_{A \oplus A^*}) = T_{\mu+\gamma} \mathcal{N}$, and therefore Theorem 4.8 is also valid in the slightly more general case of $\mathcal{N} + \kappa \text{Id}_E$, where \mathcal{N} is skew-symmetric and κ is a scalar. But there is no analogous statement for more general non skew-symmetric endomorphisms of $A \oplus A^*$. (In theorem 4.1 of [8], because of a change of notation in the course of the proof, the assumption $N^2 = \lambda_2$ should be replaced by $(N - \lambda_1)^2 = \lambda_2$. With this modification, that theorem is equivalent to the preceding generalized form of Theorem 4.8.)

We obtain the following converse of Theorem 4.8 as a particular case of Theorem 3.7.

Theorem 4.9 *Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid such that $A \oplus A^*$ is an irreducible Courant algebroid. If $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^t N \end{pmatrix}$ is a Nijenhuis tensor for $A \oplus A^*$, then N is proportional to a cps structure on A , and ${}^t N$ is proportional to a cps structure on A^* .*

We now outline an alternate, computational proof of Theorem 4.5 that does not use the Poisson bracket of \mathcal{A} . This longer proof consists of computing the vector part and the form part of $(T_{\mu+\gamma} \mathcal{N})(X, Y)$, $(T_{\mu+\gamma} \mathcal{N})(\xi, \eta)$, $(T_{\mu+\gamma} \mathcal{N})(X, \eta)$ and $(T_{\mu+\gamma} \mathcal{N})(\xi, Y)$, and then the duality product of each with Z or ζ . It utilizes the definitions

$$[X, Y] = [X, Y]^\mu, [X, \eta] = -i_\eta d_\gamma X + \mathcal{L}_X^\mu \eta, [\xi, Y] = \mathcal{L}_\xi^\gamma Y - i_Y d_\mu \xi, [\xi, \eta] = [\xi, \eta]^\gamma,$$

and $\mathcal{N}X = NX$, $\mathcal{N}\xi = -{}^t N\xi$. Clearly

$$\langle (T_{\mu+\gamma} \mathcal{N})(X, Y), Z + \zeta \rangle = \langle (T_\mu N)(X, Y), \zeta \rangle \quad (44)$$

and

$$\langle (T_{\mu+\gamma} \mathcal{N})(\xi, \eta), Z + \zeta \rangle = \langle (T_\gamma {}^t N)(\xi, \eta), Z \rangle. \quad (45)$$

One finds, after a computation,

$$\langle (T_{\mu+\gamma} \mathcal{N})(X, \eta), Z \rangle = \langle (T_\mu N)(Z, X) + [N^2 Z, X]^\mu - N^2 [Z, X]^\mu, \eta \rangle, \quad (46)$$

$$\langle (T_{\mu+\gamma}\mathcal{N})(X, \eta), \zeta \rangle = \langle (T_\gamma {}^tN)(\eta, \zeta), X \rangle + d_\gamma(N^2X)(\eta, \zeta) - (d_\gamma X)(\eta, {}^tN^2\zeta). \quad (47)$$

Similarly, one finds

$$\langle (T_{\mu+\gamma}\mathcal{N})(\xi, Y), \zeta \rangle = \langle (T_\gamma {}^tN)(\zeta, \xi) + [{}^tN^2\zeta, \xi]^\gamma - {}^tN^2[\zeta, \xi]^\gamma, Y \rangle, \quad (48)$$

$$\langle (T_{\mu+\gamma}\mathcal{N})(\xi, Y), Z \rangle = \langle (T_\mu N)(Y, Z), \xi \rangle + d_\mu({}^tN^2\xi)(Y, Z) - (d_\mu\xi)(Y, N^2Z). \quad (49)$$

If condition $N^2 = \lambda \text{Id}_A$ is satisfied, equations (46), (47), (48) and (49) simplify and we recover the result of Theorem 4.5.

Remark 4.10 From equations (44), (45), (46) and (48), we see that the conclusion of Theorem 4.9 is valid when (A, μ) or (A^*, γ) is an irreducible Lie algebroid.

4.6 Deformations of Lie bialgebroids

When (A, μ) and (A^*, γ) are Lie algebroids, if the torsions of $T_\mu N$ and $T_\gamma {}^tN$ vanish, $\mu_N = \{N, \mu\}$ and $\gamma_{-{}^tN} = \{N, \gamma\}$ are Lie algebroid structures on A and A^* , respectively. By (40), $\{\tilde{\mathcal{N}}, \tilde{\mu} + \tilde{\gamma}\} = \{N, \mu\} + \{N, \gamma\}$, therefore ‘deforming’ by \mathcal{N} the Dorfman bracket of the double $A \oplus A^*$, equipped with Courant algebroid structure $\Theta = \mu + \gamma$, amounts to considering the ‘double’ of the pair of Lie algebroids (A, μ_N) and $(A^*, \gamma_{-{}^tN})$. However the Lie algebroids (A, μ_N) and $(A^*, \gamma_{-{}^tN})$ do not in general constitute a Lie bialgebroid.

Theorem 4.11 *Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid. Assume that N is proportional to a cps structure on A and that tN is proportional to a cps structure on A^* . Then $((A, \mu_N), (A^*, \gamma_{-{}^tN}))$ is a Lie bialgebroid and its double is the Courant algebroid $(A \oplus A^*, \{\tilde{\mathcal{N}}, \mu + \gamma\})$.*

Proof This result is a corollary of Proposition 4.2 and Theorem 4.8. \square

Remark 4.12 It is possible to consider the deformation of a Lie bialgebroid by a pair of unrelated vector bundle endomorphisms, $N : A \rightarrow A$ and $N' : A^* \rightarrow A^*$, satisfying $T_\mu N = 0$ and $T_\gamma N' = 0$. The condition for the pair of Lie algebroids $(A, \{N, \mu\})$ and $(A^*, \{N', \gamma\})$ to constitute a Lie bialgebroid is

$$\{\{N, \mu\} + \{N', \gamma\}, \{N, \mu\} + \{N', \gamma\}\} = 0. \quad (50)$$

Given that $\{\{N, \mu\}, \{N, \mu\}\} = 0$ and $\{\{N', \gamma\}, \{N', \gamma\}\} = 0$, this condition becomes

$$\{\{N, \mu\}, \{N', \gamma\}\} = 0. \quad (51)$$

This compatibility condition, $\{\mu_N, \gamma_{N'}\} = 0$, means that each deformed structure, μ_N and $\gamma_{N'}$, is a cocycle for the other, or equivalently, that d_{μ_N} is a derivation of $[\ , \]_{\gamma_{N'}}$, or that $d_{\gamma_{N'}}$ is a derivation of $[\ , \]_{\mu_N}$. (If $N' = \text{Id}_{A^*}$, then condition (51) means that d_{μ_N} is a derivation of $[\ , \]_{\gamma}$, or d_{γ} is a derivation of $[\ , \]_{\mu_N}$. This result is in [8], theorem 3.1.)

4.7 Deformations of trivial Lie bialgebroids

We now consider the particular case of the trivial Lie bialgebroids, such as the generalized tangent bundles. It follows from Theorem 4.5 that, if $((A, \mu), (A^*, 0))$ is the trivial Lie bialgebroid associated with the Lie algebroid (A, μ) , then

$$\widetilde{T_{\mu}\mathcal{N}} = \widetilde{T_{\mu}N}. \quad (52)$$

In particular, in the case of a trivial Lie bialgebroid $((A, \mu), (A^*, 0))$, deforming the Dorfman bracket of the double by \mathcal{N} amounts to deforming (A, μ) by N , and Proposition 4.2, Theorem 4.8 and Remark 4.10 imply the following.

Corollary 4.13 *Let (A, μ) be a Lie algebroid, and let N be a vector bundle endomorphism of A . Let $[\ , \]$ be the Dorfman bracket of the double of the*

trivial Lie bialgebroid $((A, \mu), (A^, 0))$, and let $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -t_N \end{pmatrix}$.*

- (i) The deformed bracket $[\ , \]_{\mathcal{N}}$ is the double of the bracket $[\ , \]_N^{\mu}$.*
- (ii) If $T_{\mu}N$ vanishes, then $((A, \mu_N), (A^*, 0))$ is a trivial Lie bialgebroid.*
- (iii) If N is proportional to a cps structure on A , then the torsion of \mathcal{N} vanishes.*
- (iv) Conversely, if the torsion of \mathcal{N} vanishes, and if A is irreducible, then N is proportional to a cps structure on A .*
- (v) If N is proportional to a cps structure on A , then the double of the trivial Lie bialgebroid $((A, \mu_N), (A^*, 0))$ is the Courant algebroid $(A \oplus A^*, \{\tilde{\mathcal{N}}, \mu\})$.*

For the case of a generalized tangent bundle, $TM \oplus T^*M$, parts (i) and (ii) of Corollary 4.13 were proved in theorems 2 and 3 of [7]. It was also proved in theorem 3 that, when the base manifold M is connected, if the torsions of N and \mathcal{N} both vanish, then N is proportional to a cps structure on TM . Since, by lemma 2 of [7], a tangent bundle over a connected base is an irreducible Lie algebroid, this result is implied by (iv) above.

There is a more interesting result that does not require N^2 to be a scalar multiple of the identity.

Theorem 4.14 *Let (A, μ) be a Lie algebroid, and let N be a vector bundle endomorphism of A . If N is a Nijenhuis tensor for (A, μ) , then $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^tN \end{pmatrix}$ is a weak deforming tensor for the Courant algebroid $(A \oplus A^*, \mu)$, and $\{\tilde{\mathcal{N}}, \mu\}$ is a Courant algebroid structure on $A \oplus A^*$, which is the double of the trivial Lie bialgebroid defined by (A, μ_N) .*

Proof The hypothesis $T_\mu N = 0$ is equivalent to $\{\{N, \mu\}, N\} = \{\mu, N^2\}$. Because $\{\mu, \mu\} = 0$, this relation implies that $\{\{N, \mu\}, N\}$ is a d_μ -cocycle and therefore that $\{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\}$ is a d_μ -cocycle. Therefore \mathcal{N} is a weak deforming tensor for μ . The Courant algebroid structure $\{\tilde{\mathcal{N}}, \mu\}$ is then $\{N, \mu\}$, i.e., the double of the trivial Lie bialgebroid $((A, \mu_N), (A^*, 0))$. \square

4.8 Compatible structures and deforming tensors

We shall show that various types of composite structures on Lie algebroids, for which see, e.g., [21] [22] and references cited there, give rise to infinitesimal deformations of the Dorfman bracket of the double of any trivial Lie bialgebroid. We assume that (A, μ) is a Lie algebroid, and we consider the trivial Lie bialgebroid $((A, \mu), (A^*, 0))$,

Proposition 4.15 *Let N be a vector bundle endomorphism of A , and let π be a bivector on A such that $N\pi = \pi {}^tN$. If (π, N) is a PN-structure on A , then the skew-symmetric endomorphism of $A \oplus A^*$, $\mathcal{N} = \begin{pmatrix} N & \pi \\ 0 & -{}^tN \end{pmatrix}$ is a weak deforming tensor for $(A \oplus A^*, \mu)$.*

Proof We denote by $C_\mu(\pi, N) = \{\pi, \{N, \mu\}\} + \{N, \{\pi, \mu\}\}$ the tensor whose vanishing expresses the compatibility of a Poisson structure π and a Nijenhuis tensor N on A . We compute

$$\begin{aligned} \{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\} &= \{\{N + \pi, \mu\}, N + \pi\} \\ &= \{\{N, \mu\}, N\} + \{\{\pi, \mu\}, \pi\} + \{\{N, \mu\}, \pi\} + \{\{\pi, \mu\}, N\} \\ &= \{\{N, \mu\}, N\} + [\pi, \pi]^\mu - C_\mu(\pi, N). \end{aligned}$$

Here $[\ , \]^\mu$ is the Schouten-Nijenhuis bracket of multivectors. Therefore, if we assume that π is a Poisson bivector and that N and π are compatible,

then $\{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\} = \{\{N, \mu\}, N\}$. When N is a Nijenhuis tensor on (A, μ) , $\{\{N, \mu\}, N\}$ is a d_μ -cocycle and therefore $\{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\}$ is a d_μ -cocycle. \square

As a consequence we recover the well-known fact that when (π, N) is a PN-structure on A , then $\{\tilde{\mathcal{N}}, \mu\} = \{N, \mu\} + \{\pi, \mu\}$ is a Courant algebroid structure on $A \oplus A^*$, the double of the Lie bialgebroid $((A, \mu_N), (A^*, \gamma_\pi))$, where $\gamma_\pi = \{\pi, \mu\}$. See, e.g., theorem 4 of [22].

If N^2 is proportional to the identity of A and if π is a bivector such that $N\pi = \pi {}^tN$, then \mathcal{N}^2 is proportional to the identity of $A \oplus A^*$ and $\widetilde{T_\mu(\mathcal{N})}$ is identified with $T_\mu(N) - \frac{1}{2}[\pi, \pi]^\mu + \frac{1}{2}C_\mu(\pi, N)$ in \mathcal{A}^3 . Using the bigrading of \mathcal{A} , we conclude,

Proposition 4.16 *If N is proportional to an almost cps structure on A and π is a bivector such that $N\pi = \pi {}^tN$, then $T_\mu(\mathcal{N}) = 0$ if and only if (π, N) is a PN-structure.*

We can also relate ΩN -structures with deforming tensors, obtaining an analogue of Proposition 4.15, although there is no obvious analogue of Proposition 4.16.

Proposition 4.17 *Let N be a vector bundle endomorphism of A , and let ω be a 2-form on A such that $\omega N = {}^tN\omega$. If (ω, N) is an ΩN -structure on A , then the skew-symmetric endomorphism of $A \oplus A^*$, $\mathcal{N} = \begin{pmatrix} N & 0 \\ \omega & -{}^tN \end{pmatrix}$ is a weak deforming tensor for $(A \oplus A^*, \mu)$.*

Proof We compute

$$\begin{aligned} \{\{\{\tilde{\mathcal{N}}, \mu\}, \tilde{\mathcal{N}}\} &= \{\{N + \omega, \mu\}, N + \omega\} \\ &= \{\{N, \mu\}, N\} + \{\{N, \mu\}, \omega\} + \{\{\omega, \mu\}, N\} \end{aligned}$$

since $\{\{\omega, \mu\}, \omega\} = 0$. When (ω, N) is an ΩN -structure, both $d_\mu\omega = \{\mu, \omega\}$ and $d_{\mu_N}\omega = \{\{N, \mu\}, \omega\}$ vanish. We conclude, using the vanishing of the torsion of N , as in the proof of Proposition 4.15. \square

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